Solitary Waves and Compactons in a class of Generalized Korteweg-DeVries Equations

Fred Cooper

Theoretical Division, Los Alamos National Laboratory,

Los Alamos, NM 87545

and

Physics Department, University of New Hampshire,

Durham, NH 03824

Harvey Shepard

Physics Department, University of New Hampshire,

Durham, NH 03824

Pasquale Sodano

Dipartimento di Fisica and Sezione, INFN Universita di Perugia,

06100 Perugia, Italy

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Abstract

We study the class of generalized Korteweg-DeVries equations derivable from the Lagrangian: $L(l,p) = \int \left(\frac{1}{2}\varphi_x\varphi_t - \frac{(\varphi_x)^l}{l(l-1)} + \alpha(\varphi_x)^p(\varphi_{xx})^2 \right) dx$, where the usual fields $u(x,t)$ of the generalized KdV equation are defined by $u(x,t) = \varphi_x(x,t)$. This class contains compactons, which are solitary waves with compact support, and when $l = p+2$, these solutions have the feature that their width is independent of the amplitude. We consider the Hamiltonian structure and integrability

properties of this class of KdV equations. We show that many of the properties of the solitary waves and compactons are easily obtained using a variational method based on the principle of least action. Using a class of trial variational functions of the form $u(x,t) = A(t)\exp\left[-\beta(t)\left|x-q(t)\right|^{2n}\right]$ we find soliton-like solutions for all n, moving with fixed shape and constant velocity, c. We show that the velocity, mass, and energy of the variational travelling wave solutions are related by $c = 2rEM^{-1}$, where $r = (p+l+2)/(p+6-l)$, independent of n.

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1 Introduction

Recently, Rosenau and Hyman [1] have shown that in a particular generalization of the KdV equation, defined by parameters (m,n), namely

$$K(n,m) : u_t + (u^m)_x + (u^n)_{xxx} = 0, \tag{1}$$

that a new form of solitary wave with compact support and width independent of amplitude exists. For their choice of generalized KdV equations the compactons with $m = n \leq 3$ had the form $[cos(a\xi)]^{2/(m-1)}$, where $\xi = x - ct$ and for m=2,3 they obtained:

$$\begin{aligned} &K(2,2) \qquad u_c = \frac{4c}{3}\cos^2(\xi/4) \\ &K(3,3) \qquad u_c = \left(\frac{3c}{2}\right)^{1/2}\cos(\xi/3). \end{aligned} \tag{2}$$

Unlike the ordinary KdV equation, the generalized KdV equation considered by Rosenau and Hyman was not derivable from a first order Lagrangian except for $n = 1$, and did not possess the usual conservation laws of energy and mass that the KdV equation possessed. It is presumed that the generalized KdV equations found by the above authors are not completely integrable, but instead possess only a finite number of conservation laws. Because of this, we were led to consider a different generalization of the KdV equation based on a first order Lagrangian formulation. That is, we consider

$$L(l,p) = \int \left(\frac{1}{2}\varphi_x \varphi_t - \frac{(\varphi_x)^l}{l(l-1)} + \alpha(\varphi_x)^p(\varphi_{xx})^2 \right) dx. \tag{3}$$

This Lagrangian leads to a generalized sequence of KdV equations of the form:

$$K^*(l,p) : u_t = u_x u^{l-2} + \alpha \left(2u_{xxx}u^p + 4pu^{p-1}u_x u_{xx} + p(p-1)u^{p-2}(u_x)^3 \right) \tag{4}$$

where

$$u(x) = \varphi_x(x). \tag{5}$$

These equations have the same terms as the equations considered by Rosenau and Hyman, but the relative weights of the terms are quite different leading to the possibility that the integrability properties might be different. [For the purposes of comparison it may be helpful to note that their set (m, n) corresponds to our $(l-1, p+1)$.]The rest of the paper is organized follows: In section 2 we discuss some exact travelling wave solutions to (4). In section 3 we derive the conservation laws and discuss the Hamiltonian structure of these equations. In section 4 we apply the time dependent variational approach to obtaining approximate solitary wave solutions, and in section 5 we compare the variational solutions to the exact ones.

2 Exact solitary wave and compacton solutions

If we assume a solution to (4) in the form of a travelling wave:

$$u(x,t) = f(\xi) = f(x+ct), \tag{6}$$

one obtains for f:

$$cf' = f' f^{l-2} + \alpha \left(2f''' f^p + 4pf^{p-1} f' f'' + p(p-1) f^{p-2} f'^3 \right). \tag{7}$$

Integrating twice we obtain:

$$\frac{c}{2} f^2 - \frac{f^l}{l(l-1)} - \alpha f'^2 f^p = C_1 f + C_2. \tag{8}$$

We seek solutions where the integration constants, C_1 and C_2 are zero. This puts lower bounds on l and p: $l > 1$ and $f''f^p \to 0, f'^2 f^{p-1} \to 0$ at edges where $f \to 0$. Then we obtain

$$\alpha f'^2 = \frac{c}{2} f^{2-p} - \frac{f^{l-p}}{l(l-1)}. \tag{9}$$

For finite f' at the edges, we must have $p \leq 2, l \geq p$.

Let us now look at some special examples. (Note that we have chosen signs so that all travelling waves have $u > 0$ and move to the left.) The usual KdV equation has $\alpha = 1/2$, $l = 3$, $p = 0$. For that case one has the well known soliton:

$$u = (3c)\mathrm{sech}^2 \left[\sqrt{3\mathrm{c}/2}\xi \right]. \tag{10}$$

We define the “mass” M via

$$M = \int_{-\infty}^{\infty} dx [u(x,t)]^2. \tag{11}$$

For this solution we find that we can express M and E in terms of c as follows: $M = 24c^{3/2}$, $E = \frac{36}{5}c^{5/2}$ so that

$$c = \frac{10}{3} E M^{-1} = (M/24)^{2/3}. \tag{12}$$

The case $l = p + 2$ is the case relevant for compactons whose width is independent of the velocity c. For $p = 1$, $\alpha = 1/2$ one obtains the compacton solution:

$$u_1 = 3c \cos^2(\xi/\sqrt{12}), \tag{13}$$

where $|\xi| \leq \sqrt{3}\pi$. One finds: $M = \frac{27}{4}\pi\sqrt{3}c^2$, $E = \frac{27}{8}\sqrt{3}\pi c^3$ so that

$$c = 2EM^{-1} = \left(\frac{4M}{27\pi\sqrt{3}} \right)^{1/2}. \tag{14}$$

There is another compacton solution with $p = 2$, $\alpha = 3$.

$$u_2 = \sqrt{6c}\cos(\xi/6) \tag{15}$$

with $|\xi| \leq 3\pi$. For this compacton, one finds $M = 18\pi c$, $E = \frac{9\pi c^2}{2}$ so that

$$c = 4EM^{-1} = \frac{M}{18\pi}. \tag{16}$$

For the values, $l = 3$, $p = 2$ there is a compacton whose width depends on the velocity . Choosing $\alpha = 1/4$ we find:

$$u = 3c - (\xi^2)/6 \tag{17}$$

on the interval

$$|\xi| \leq 3\sqrt{2c}; \tag{18}$$

elsewhere it is zero. For this compacton one finds: $M = \frac{144}{5}\sqrt{2}c^{5/2}$, $E = \frac{72}{7}\sqrt{2}c^{7/2}$ so

$$c = \frac{14}{5}EM^{-1} = \left(\frac{5M}{144\sqrt{2}}\right)^{2/5}. \tag{19}$$

Thus, apart from constants we find the same functional form for the compactons for our generalized KdV equations as those found by Rosenau and Hyman in their different generalization of the KdV equation.

3 Conservation laws and canonical structure

Equation (4) can be written in canonical form displaying the same Poisson bracket structure as found for the KdV equation:

$$u_t = \partial_x \frac{\delta H}{\delta u} = \{u, H\} \tag{20}$$

where H is the Hamiltonian obtained from the Lagrangian (3),

$$\begin{aligned} H &= \int [(\pi\dot{\varphi}) - L]\, dx \\ &= \int \left[\frac{(\varphi_x)^l}{l(l-1)} - \alpha(\varphi_x)^p(\varphi_{xx})^2\right] dx, \\ &= \int \left[\frac{u^l}{l(l-1)} - \alpha u^p (u_x)^2\right] dx. \qquad (21) \\ & \qquad (22) \end{aligned}$$

By the usual arguments [2] this is consistent with a Poisson bracket structure

$$\{u(x), u(y)\} = \partial_x \delta(x - y). \tag{23}$$

Let us now show that we have a system of equations which have exactly the same first three conservation laws as the ordinary KdV equation, namely the area, mass and energy. This is unlike the equations studied by Rosenau and Hyman that did not conserve the mass and energy, but instead had different conserved quantities.

We have

$$u_t = \partial_x \frac{\delta H}{\delta u} \tag{24}$$

so that the "area" under $u(x,t)$ is conserved:

$$l \int u(x,t) dx \equiv H_0 \tag{25}$$

Multiplying by $u(x,t)$ we find:

$$\partial_t(\frac{u^2}{2}) = \partial_x \left[\frac{u^l}{l} + \alpha\{(p-1)u^p u_x^2 + 2u^{p+1}u_{xx}\} \right] \tag{26}$$

which leads to the conservation of "mass"

$$(1/2) \int u^2(x,t) dx = (1/2)M \equiv H_1 \tag{27}$$

For the KdV equation H_1 was a second Hamiltonian under a second Poisson bracket structure. From Lagrange's equations we immediately get a third conservation law, the energy:

$$H = \int \left[\frac{u^l}{l(l-1)} - \alpha u^p (u_x)^2 \right] dx \equiv H_2. \tag{28}$$

The energy provided the first Poisson bracket structure: Considering the mass as a second Hamiltonian, the KdV equation has a second Poisson bracket structure using H_1. Assuming

$$u_t = \{u, H_1\} = \int dy \{u(x), u(y)\}_1 \frac{\delta H_1}{\delta u(y)}, \tag{29}$$

one finds for the KdV equation that

$$\{u(x), u(y)\}_1 = \left(D^3 + \frac{1}{3}(Du + uD) \right) \delta(x-y) \tag{30}$$

where $D = \partial_x$. With this assumed Poisson bracket structure one again recovers the KdV equation. This Poisson bracket structure is identical to the Virasoro algebra with a specific central charge. This fact enables one to show that there is an infinite number of conservation laws in the KdV equation, and it is an exactly integrable system [2].

For the generalized KdV equations we find that we can write

$$u_t = \left(\alpha(D^2 u^p D + D u^p D^2) + \frac{1}{l}(D u^{l-2} + u^{l-2} D)\right) u \tag{31}$$

so that there is a chance for a second Hamiltonian if the Jacobi identity is satisfied. One can postulate that the second Poisson bracket structure is given by

$$\{u(x), u(y)\}_1 = \left(\alpha(D^2 u^p D + D u^p D^2) + \frac{1}{l}(D u^{l-2} + u^{l-2} D)\right) \delta(x-y). \tag{32}$$

So we need to show for what l, p this bracket structure obeys the Jacobi Identity,where the bracket is defined by :

$$\{F[u], G[u]\} = \int_{-\infty}^{\infty} dx dy \frac{\delta F}{\delta u(x)} \{u(x), u(y)\}_1 \frac{\delta G}{\delta u(y)}. \tag{33}$$

One can show immediately that the Hamiltonians H_1 and H_2 commute using either Poisson Bracket structure (23) or (32).

We have that

$$\{H_2[u], H_1[u]\} = \int_{-\infty}^{\infty} dx dy \frac{\delta H_2}{\delta u(x)} \{u(x), u(y)\}_1 \frac{\delta H_1}{\delta u(y)}. \tag{34}$$

For the usual bracket structure (23) we can rewrite (34) as

$$\begin{aligned} \{H_2[u], H_1[u]\} &= \int_{-\infty}^{\infty} dx u_t(x) \left(\frac{\delta H_1}{\delta u(x)}\right) \\ &= \frac{1}{2}\partial_t \int_{-\infty}^{\infty} dx u^2(x,t) = 0. \end{aligned} \tag{35}$$

For the second bracket structure (32) we have instead:

$$\{H_2[u], H_1[u]\}_1 = \int_{-\infty}^{\infty} dx u_t(x) \left(\frac{\delta H_2}{\delta u(x)}\right)$$

$$
\begin{aligned}
&= \int_{-\infty}^{\infty} dx \frac{\delta H_2}{\delta u(x)} \partial_x \left(\frac{\delta H_2}{\delta u(x)} \right) \\
&= \frac{1}{2} \int_{-\infty}^{\infty} dx \partial_x \left(\frac{\delta H_2}{\delta u(x)} \right)^2 = 0. \qquad (36)
\end{aligned}
$$

Encouraged by this result we have attempted to repeat the induction proof of the existence of an infinite number of conservation laws, assuming as in the KdV equation that one has the conservation laws obey the recursion relations:

$$
\left(\alpha (D^2 u^p D + D u^p D^2) + \frac{1}{l} (D u^{l-2} + u^{l-2} D) \right) \frac{\delta H_{n-1}}{\delta u(x)} = D \frac{\delta H_n}{\delta u(x)}. \qquad (37)
$$

Starting with H_0 defined by (25) we get the candidate Hamiltonian:

$$
H_1 = \int \frac{u^{l-1}(x,t)}{(l-1)} dx \qquad (38)
$$

instead of (27). If we now ask if this is conserved by considering the equation

$$
\frac{dH_1}{dt} = \{H_1, H_2\} \qquad (39)
$$

using the first Poisson bracket structure, we find that the right hand side of (39) is not a total divergence unless $l = 3$. For $l = 3$ one has :

$$
\left(\alpha (D^2 u^p D + D u^p D^2) + \frac{1}{l} (D u^{l-2} + u^{l-2} D) \right) \frac{\delta H_1}{\delta u(x)} = D \frac{\delta H_2}{\delta u(x)} \qquad (40)
$$

However if we iterate one more time (with $l = 3$) we obtain:

$$
\left(\alpha (D^2 u^p D + D u^p D^2) + \frac{1}{3} (D u + u D) \right) \frac{\delta H_2}{\delta u(x)} = D F_3(x) \qquad (41)
$$

and we find by explicit construction that $F_3(x)$ is not the variational derivative of a local Hamiltonian unless $p = 0$. Thus this bi-Hamiltonian method of finding an infinite number of conservation laws only works for the original KdV equation. We surmise that (32) is not a valid bracket structure and that

$$\{H[u],\{F[u],G[u]\}\}+\{G[u],\{H[u],F[u]\}\}+\{F[u],\{G[u],H[u]\}\}=0 \quad (42)$$

is *not* satisfied for the postulated second bracket. Thus we have not succeeded in showing that these new equations are exactly integrable, and we are in the same situation, in spite of having a first order Lagrangian, as for the generalized KdV equations of Rosenau and Hyman [1].

We have not as yet performed numerical simulations of the scattering of our new compacton solutions. For Rosenau and Hyman such numerical experiments produced behavior very similar to but not exactly the same as that observed in completely integrable systems, namely, stability and preservation of shape. They find that elastic collisions are accompanied by the production of low amplitude compacton-anticompacton pairs [1].

4 Variational approach

Our time-dependent variational approach for studying solitary waves is related to Dirac's variational approach to the Schrodinger equation [5], [6]. In our previous work [3] [4], we introduced a post-Gaussian variational approximation, a continuous family of trial variational functions more general than Gaussians, which can still be treated analytically. Assuming a variational ansatz of the form $u(x,t) = A(t)\exp\left[-\beta(t)\left|x-q(t)\right|^{2n}\right]$,we will extremize the effective action for the trial wave functional and determine the classical dynamics for the variational parameters. We will find that for all (l,p) the dynamics of the variational parameters lead to solitary waves moving with constant velocity and constant amplitude. For the special case of $l = p+2$ we find immediately that the width of the soliton is independent of the amplitude and velocity. Correct functional relations between energy, mass, amplitude and velocity are obtained very quickly from the variational method, although one does not find that the $l = p+2$ variational solitons have compact support. We will find that most of the properties of the single "soliton" solutions to these equations can be obtained by using this very simple trial wave function ansatz and extremizing the action.

The starting point for the variational calculation is the action

$$\Gamma = \int L dt, \quad (43)$$

where L is given by (3).

Just as we did in our study of the KdV equation we choose a trial wave function of the form:

$$u_v(x,t) = A(t)\exp\left[-\beta(t)|x-q(t)|^{2n}\right], \tag{44}$$

where n is an arbitrary continuous, real parameter.

The variational parameters have a simple interpretation in terms of expectation values with respect to the "probability"

$$P(x,t) = \frac{\left[u_v(x,t)\right]^2}{M(t)}, \tag{45}$$

where the mass M is defined as above

$$M(t) \equiv \int \left[u_v(x,t)\right]^2 dx. \tag{46}$$

(Here we allow M to be a function of t, even though M is conserved)

Since $\langle x - q(t)\rangle = 0$, $q(t) = \langle x\rangle$. From (46) and (44) we have

$$A(t) = \frac{M^{1/2}(2\beta)^{1/4n}}{\left[2\Gamma\left(\frac{1}{2n}+1\right)\right]^{1/2}}. \tag{47}$$

The inverse width β is related to

$$G_{2n} \equiv \langle |x-q(t)|^{2n}\rangle = \frac{1}{4n\beta}. \tag{48}$$

Following our approach in [4], we find that the action for the trial wave function (44) is given by:

$$\begin{aligned}\Gamma(q,\beta,M,n) &= \int\left(-\frac{1}{2}M\dot{q} - C_1(n)\beta^{(l-2)/4n}M^{l/2} + C_2(n)M^{1+p/2}\beta^{(p+4)/4n}\right)dt \\ &\equiv \int L_1(q,\dot{q},M,\beta)dt, \end{aligned} \tag{49}$$

where

$$
\begin{aligned}
C_1(n) &= \frac{1}{l(l-1)}\left(\frac{2^l}{l^2}\right)^{1/4n}\left[2\Gamma\left(\frac{1}{2n}+1\right)\right]^{(2-l)/2} \\
C_2(n) &= 4\alpha n(2)^{(p+2)/4n}(2+p)^{\frac{1}{2n}-2}\frac{\Gamma\left(2-\frac{1}{2n}\right)}{[2\Gamma\left(\frac{1}{2n}+1\right)]^{1+p/2}}.
\end{aligned} \tag{50}
$$

We eliminate the variable of constraint β (using $\delta\Gamma/\delta\beta = 0$) and find

$$\beta = [d(n)]^{4n}\, M^{2n(p+2-l)/(l-p-6)}, \tag{51}$$

where

$$d(n) = \left[\frac{(p+4)C_2(n)}{(l-2)C_1(n)}\right]^{1/(l-p-6)}. \tag{52}$$

From (51) we see that when

$$l = p+2, \tag{53}$$

the width of the soliton β does not depend on M and thus is independent of the amplitude or velocity. This special case is precisely the case when the exact solution is a compacton.

We now eliminate β in favor of M, and symmetrizing the Lagrangian (3) we obtain [7]:

$$L_2 = \frac{1}{4}\left(q\dot{M} - \dot{q}M\right) - H(M), \tag{54}$$

where

$$H(M) = \left(C_1 d^{(l-2)} - C_2 d^{(p+4)}\right) M^r, \tag{55}$$

where $r = (p+l+2)/(p+6-l)$. Extremizing the action yields:

$$
\begin{aligned}
\dot{M} = 0 \;&\Longrightarrow\; M = \text{const.} \\
&\Longrightarrow\; \beta = \text{const.}
\end{aligned} \tag{56}
$$

and

$$\dot{q} = -2r\left(C_1 d^{(l-2)} - C_2 d^{(p+4)}\right) M^{r-1}, \tag{57}$$

as well as a conserved energy

$$E = \left(C_1 d^{(l-2)} - C_2 d^{(p+4)}\right) M^r \tag{58}$$

Thus the velocity of the solitary wave is constant and can be related to the conserved energy via

$$\dot{q} = -c = -2rEM^{-1}. \tag{59}$$

This is precisely the form we obtained for the exact solutions.

We have not yet extremized the action with respect to the variational parameter n which is equivalent to extremizing the energy with respect to n. We perform this extremization graphically for each value of l, p. The explicit form of the trial wave function is:

$$\begin{aligned} u_v(x,t) &= d[n,p,l] M^{2/(p+6-l)} 2^{1/4n} \left[2\Gamma\left(1+\frac{1}{2n}\right)\right]^{-1/2} \times \\ &\quad \exp\left[-d^{4n} M^{2n(l-p-2)/(p+6-l)} |x+ct-x_0|^{2n}\right], \end{aligned} \tag{60}$$

where d is given by (52).

Now let us see how these trial wave functions compare with the exact answers for special cases. Since we explicitly know the M dependence of the answer, we can set M=1 as our normalization for both the variational and exact solitons.

First let us review the results for the KdV equation: Here the variational wave function is obtained by first setting $p = 0, \alpha = 1/2$ and $l = 3$. One then extremizes the action in the constant parameter n. We find numerically that n=.877, which also extremizes the energy to be $.035999 M^{5/3}$ and determines the velocity to be $.119995 M^{2/3}$. In figure 1a and 1b we compare this variational result to the exact soliton given by:

$$u = (3c)\mathrm{sech}^2\left[\sqrt{3\mathrm{c}/2}(\mathrm{x}+\mathrm{ct})\right]. \tag{61}$$

We see from fig 1b that globally we achieve an accuracy of better than 1%. For this solution the velocity and energy are:

$$c = (M/24)^{2/3} = .120187 M^{2/3} \tag{62}$$

$$E = \frac{3}{10} M c = .0360562 M^{5/3}. \tag{63}$$

Thus we obtain the velocity (and the energy) accurate to 0.2% from the variational calculation.

Next let us look at the compacton that is a segment of a parabola (17). Here the variational wave function is obtained by setting $p = 2, \alpha = 1/4$ and $l = 3$. Mininimizing the action in the constant parameter n we find numerically that n=1.423, which also extremizes the energy to be .0803831 and determines the velocity to be .225073 In figure 2a and 2b we compare this variational result to the exact soliton given by:

$$u = 3c - \frac{(x+ct)^2}{6} \tag{64}$$

on the interval

$$|\xi| \leq 3\sqrt{2c} \tag{65}$$

otherwise zero. We notice that the global accuracy is a few per cent except near the place where the true compacton goes to zero. For this compacton one finds:

$$c = \left(\frac{5M}{144\sqrt{2}} \right)^{2/5} = .227006 M^{2/5} \tag{66}$$

and

$$E = 5/14 M c = .0810735 M^{7/5} \tag{67}$$

Thus we find that the velocity (and the energy) are determined to 0.8% accuracy.

Next let us look at the compacton given by (13). Here the variational wave function is obtained by setting $p = 1, \alpha = 1/2$ and $l = 3$. Extremizing the action in the constant parameter n we find numerically that n=1.154, which also extremizes the energy to be .054888 and determines the velocity to be .164666 In figure 3a and 3b we compare this variational result to the exact soliton given by:

$$u_1 = 3c \cos^2(\xi/\sqrt{12}), \tag{68}$$

where $\xi \leq \sqrt{3}\pi$

We notice that the global accuracy is a few per cent except near the place where the true compacton goes to zero. For this compacton one finds:

$$c = \left(\frac{4M}{27\pi\sqrt{3}}\right)^{1/2} = .165003M^{1/2} \tag{69}$$

For the exact solitary wave one has the relationship:

$$E = cM/2 = .0825017M \tag{70}$$

The approximate soliton had instead:

$$E = cM/(2r) = cM/3 = .054888M \tag{71}$$

So for this compacton, the variational energy is wrong by a factor of 2/3 although the velocity is correct to 0.2%. This is the only case where the variational method does not give the exact relationship between energy and velocity. However, we note that with a change in the sign of the second term in the expression for E, eq. (58), the variational energy becomes $.0823332M$, which is accurate to 0.2%, leaving the velocity and optimal n unchanged! (Such a sign change would follow from a factor $(-1)^p$ in C_2, which would leave all other results in this work unchanged.)

Finally let us look at the compacton given by (15). Here the variational wave function is obtained by setting $p = 2, \alpha = 3$ and $l = 4$. Extremizing the action in the constant parameter n we find numerically that n=1.283, which also extremizes the energy to be .00436284 and determines the velocity to be .017451 In figure 4a and 4b we compare this variational reult to the exact soliton given by:

$$u_2 = \sqrt{6c}\cos(\xi/6). \tag{72}$$

where $\xi \leq 3\pi$

We notice that the global accuracy is a few per cent except near the place where the true compacton goes to zero. For the exact compacton one finds:

$$c = \frac{M}{18\pi} = .0176839M \tag{73}$$

The variational estimates for c and E are accurate to 1.3%. For the exact and variational soliton one has the same relationship:

$$E = cM/4 = \frac{M^2}{72\pi}. \tag{74}$$

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References

[1] P.Rosenau and J.M.Hyman, Phys. Rev. Lett. 70, 564 (1993).

[2] A. Das, Integrable Models (World Scientific Lecture Notes in Physics, Vol.30) 1989.

[3] F.Cooper, H. Shepard, C. Lucheroni, and P. Sodano, "Post-Gaussian Variational Method for the Nonlinear Schrodinger Equation: Soliton Behavior and Blowup," Physica D (to be published).

[4] F.Cooper,C. Lucheroni, H. Shepard and P. Sodano, Phys.Lett. A 173, 33 (1993).

[5] P. A. M. Dirac, Proc. Camb. Phil. Soc. 26, 376 (1930).

[6] R. Jackiw and A. Kerman, Phys. Lett. 71A, 158 (1979).

[7] L. Faddeev and R. Jackiw, Phys. Rev. Lett. 60, 1692 (1988).

FIGURE CAPTIONS

Fig. 1a u_v with $n = .876$ and u for $M = 1$ given by (61) as a function of x for the case: $p = 0$, $\alpha = 1/2$, $l = 3$.
Fig. 1b $u - u_v$ as a function of x for $p = 0$, $\alpha = 1/2$, $l = 3$.
Fig. 2a u_v with $n = 1.423$ and u for $M = 1$ given by (64) as a function of x for the case: $p = 2$, $\alpha = 1/4$, $l = 3$.
Fig. 2b $u - u_v$ as a function of x for $p = 0$, $\alpha = 1/2$, $l = 3$.
Fig. 3a u_v with $n = 1.155$ and u for $M = 1$ as a function of x given by (68) for the case: $p = 1$, $\alpha = 1/2$, $l = 3$.
Fig. 3b $u_v - u$ as a function of x for $p = 1$, $\alpha = 1/2$, $l = 3$.
Fig. 4a u_v with $n = 1.283$ and u for $M = 1$ given by (72) as a function of x for the case: $p = 2$, $\alpha = 3$, $l = 4$.

Fig. 4b $u_v - u$ as a function of x for $p = 2$, $\alpha = 3$, $l = 4$.